$$\partial\phi(\mathbf{r})/\partial n + \int_{\mathscr{s}} [\partial\phi(\boldsymbol{\rho})/\partial\mu][\partial G(\mathbf{r},\boldsymbol{\rho})/\partial n]\, d\mathscr{s} = \int_{\mathscr{s}} \phi(\boldsymbol{\rho})[\partial^2 G(\mathbf{r},\boldsymbol{\rho})/\partial n\, \partial\mu]\, d\mathscr{s} \quad (20)$$

Specialization to Uniform Flow Case

When the flow is not transonic, a thin airfoil can be treated by assuming c and $\mathbf{V}$ to be uniform everywhere, so that the two-sided surface $\mathscr{s}$ can be replaced by a single-sided surface s, having a local unit normal ν. Then, in the absence of "breathing" modes of the airfoil, the integral on the left-hand side of Eq. (20) cancels out, while in the integral on the right-hand side, the potentials can be replaced by their difference $\Delta\phi$. This is taken as positive when ϕ decreases in the positive ν direction. Equation (20) now becomes

$$\frac{\partial\phi(\mathbf{r})}{\partial n} = \int_s \Delta\phi(\mathbf{r}) \frac{\partial^2 G(\mathbf{r},\rho)}{\partial n\, \partial\nu} ds \quad (21)$$

Reversibility Relations in the Uniform Flow Case

When $\mathbf{V}$ and c are everywhere uniform, Eqs. (8) and (16) for the wave operator and its adjoint can be reduced to

$$\mathscr{L} = \nabla^2 - [(i\omega/c) - (\mathbf{V}/c)\cdot\nabla]^2 \quad (22)$$

$$\tilde{\mathscr{L}} = \nabla^2 - [(i\omega/c) + (\mathbf{V}/c)\cdot\nabla]^2 \quad (23)$$

Clearly

$$\tilde{\mathscr{L}} = \mathscr{L}^* \quad (24)$$

where $\mathscr{L}^*$ is the adjoint of $\mathscr{L}$, so that the operator is Hermetian. From another point of view, $\tilde{\mathscr{L}}$ is the wave operator for reversed flow. It can now be shown that

$$\tilde{\mathscr{L}}_{\rho}\{G(\mathbf{r},\boldsymbol{\rho})\} = \mathscr{L}_{\mathbf{r}}\{G(\mathbf{r},\boldsymbol{\rho})\} = \delta(\mathbf{r}-\boldsymbol{\rho}) \quad (25)$$

By comparison with Eq. (10) it is readily seen that G is identical to the unit source potential function ϕ_s for a uniform flow case. This has been assumed by many investigators without benefit of proof.

Conclusions

In the general case of irrotational flow, ϕ_s and G, as given by Eqs. (10) and (11b), are distinct functions, which represent disturbances spreading in opposite directions. The unit source $\phi_s(\mathbf{r},\boldsymbol{\rho})$ represents a disturbance spreading from the point $\boldsymbol{\rho}$, whereas $G(\mathbf{r},\boldsymbol{\rho})$, which appears in Eq. (20), represents a disturbance spreading from the point $\mathbf{r}$, the point at which the normal flow $\partial\phi/\partial n$ is defined. Thus the present analysis appears to conform to a concept of cause and effect according to which the normal flow resulting from a deflection of the aerodynamic surfaces causes disturbances to spread and to induce perturbations of the velocity potentials and therefore of the pressures acting on the surface at the points $\boldsymbol{\rho}$. If the unit source ϕ_s were to appear in Eq. (20), it would be necessary to assume that the induced velocity potentials anticipated the deflections of the aerodynamic surfaces, in apparent contradiction of cause and effect. It is only in the uniform flow case that this distinction disappears.

In conclusion, it will be necessary to obtain solutions to Eq. (11b) using the adjoint wave operator, in order to set up the integral equation expressed by Eq. (20). For example, if ray tracing techniques should be used to develop expressions for the function G, these would not be acoustical rays obtained from the wave equation, but rather rays satisfying the adjoint wave equation. Also, the rays would start at the point $\mathbf{r}$, where the value for $\partial\phi/\partial n$ is given, and they would proceed to the point $\boldsymbol{\rho}$, where the value for the unknown velocity potential ϕ is sought. This is in the opposite direction to the rays implied by the unit source solution ϕ_s, which originate at the point $\boldsymbol{\rho}$.

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Thin-Walled Elements in Truss Synthesis

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Introduction

In order to obtain a definitive lower-bound to the weight of optimized truss structures it is necessary to consider possible Euler and local buckling of compression members. Since critical stress for each instability mode is generally a function of a different design variable, early formulations of this inequality-constrained synthesis problem involved at least two variables per element.[1] However, it was subsequently shown[2] that imposition of simultaneous buckling in Euler and local modes allowed the formulation to be reduced to one involving a single variable and single buckling constraint per element. More recently an iterative approach, rather than a mathematical programing solution, has been used to obtain truss designs which include consideration of element stability.[3,4] Although these designs are apparently obtained by treating member areas as the only variables, the formulation seems to utilize separate constraints on Euler and local buckling stresses, and is therefore not directly related to Ref. 2.

The purpose of this Note is to present a rigorous proof of the formulation contained in Ref. 2, wherein the effect of element instability is included in the truss synthesis problem in such a way that the only required design variables are cross-sectional areas. It is shown that this formulation also leads directly to an iterative procedure which allows simple and efficient design of near-optimal fully-stressed buckling-resistant trusses.

Thin-Walled Truss Components

Consider the design of a uniform thin-walled round tubular truss member. Behavioral constraints for this element are stated as follows:

$$-\sigma_c \leqq \sigma_{ij} \leqq \sigma_T \quad (1a)$$

$$\sigma_{ij} \geqq -\pi^2 EI_i/(L_i^2 A_i) \quad (1b)$$

$$\sigma_{ij} \geqq -KE(t/D)_i \quad (1c)$$

where σ_{ij} is the stress in element i under load condition j, σ_T and σ_c are tensile and compressive yield stresses, respectively, L = length, E = modulus of elasticity, D = mean diameter, t = wall thickness, I = moment of inertia, A = cross-sectional area, and K is a buckling coefficient. It has been implied in Eqs. (1) that all elements are composed of the same material, although generalization can be accomplished in a straightforward manner, as can extension to elements of other cross-sectional shapes.

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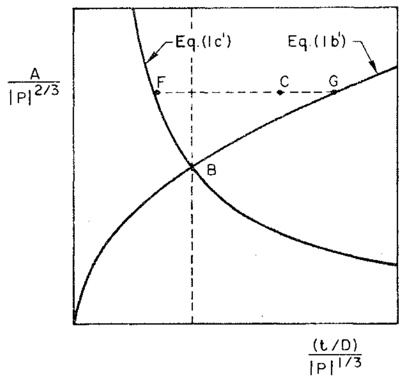

Fig. 1 Design space for thin-walled compression member.

By substituting $\sigma_{ij} = P_{ij}/A_i$, and $I \doteq \pi D^3 t/8$, Eqs. (1b) and (1c) can be written in the form

$$\frac{A_i}{|P_{ij}|^{2/3}} \geqq \left(\frac{8L_i^{\,2}}{\pi E}\right)^{1/2} \left[\frac{(t/D)_i}{|P_{ij}|^{1/3}}\right]^{1/2} \tag{1b'}$$

$$\frac{A_i}{|P_{ij}|^{2/3}} \geqq \left(\frac{1}{KE}\right) \left[\frac{(t/D)_i}{|P_{ij}|^{1/3}}\right]^{-1} \tag{1c'}$$

Equations (1b') and (1c') contain A and t/D as the two independent design variables. These equations apply only for negative (compressive) values of force P_{ij}. Relations (1b') and (1c') are illustrated schematically in Fig. 1 for an elastic truss element of specified material and length, subjected to any compressive force. It is also noted that, for any specified value of P_{ij}, the compressive yield constraint, Eq. (1a), appears as a horizontal line in Fig. 1.

The significance of this design space in connection with truss synthesis is explained as follows: suppose that at some stage in the design process a feasible, but nonoptimum, design is obtained; i.e., A_k and $(t/D)_k$ are specified for $k = 1, 2, \ldots, i, \ldots, n$, where n is the total number of truss elements. In a general indeterminate truss the values P_{ij} are functions of the A_k. Assume that the design of element i is determined by a compressive force P_{il}, where l represents a critical load condition. Any feasible design A_i, $(t/D)_i$ is represented by a point, such as C, in the unshaded region of Fig. 1. It follows that the design of element i may be altered, without changing A_i or P_{ij}, by selecting any $(t/D)_i$ which lies on horizontal line FCG. Since A_i remains constant, truss stiffness and weight are unaltered and, since this move is arbitrary and unaffected by a yield constraint, any design may be moved to a point on a vertical line through the intersection of constraints (1b') and (1c'); i.e., to the point defined by

$$\left(\frac{t}{D}\right)_i = \left(\frac{\pi}{8K^2 L_i^{\,2} E}\right)^{1/3} P_{il}^{\,1/3} \tag{2}$$

Substitution of Eq. (2) into (1b') or (1c') eliminates one design variable from the problem, and retains the same degree of freedom in the choice of A_i as existed originally. The synthesis process may now generate designs which lie anywhere along the feasible portion of the vertical line through B in Fig. 1. The form of the resulting single buckling constraint equation is either

$$\frac{A_i}{|P_{ij}|^{2/3}} \geqq \frac{2L_i^{\,2/3}}{\pi^{1/3} K^{1/3} E^{2/3}} \tag{3a}$$

or

$$\sigma_{ij} \geqq -(\pi K/8)^{1/2} (E/L_i) A_i^{\,1/2} \tag{3b}$$

Consequently, Eq. (3b) or (3a), replaces both (1b) and (1c), and synthesis may be accomplished using only A_i as the design variable. Members which satisfy the equality in (3b) in the final configuration must have (t/D) as given by Eq. (2). Members which do not satisfy the equality in Eq. (3b) may utilize any feasible $(t/D)/|P|^{1/3}$ corresponding to the value of $A/|P|^{2/3}$ for the member, where P is now a specified constant. Explicit values of t and D may be obtained by noting that $A = \pi Dt$. It may also be noted that Eq. (3b) is identical to Eq. (5) of Ref. 2.

As an alternative to synthesis based on mathematical programing the form of Eq. (3) also suggests the use of an iterative procedure for fully-stressing all truss elements. This requires that Eq. (3) and/or Eq. (1a) be satisfied as equalities. In effect, this forces all compressive members to lie at point B of their design spaces, Fig. 1. An iterative procedure applicable to the compressive elements can be constructed directly from Eq. (3a), for example, as

$$(A_i)_{p+1} = \left(\frac{2L_i^{\,2/3}}{\pi^{1/3} K^{1/3} E^{2/3}}\right) |P_{ij}|_p^{\,2/3} \tag{4}$$

where the subscript p denotes the pth value in an iterative sequence. This iterative procedure is similar in concept to that used in design of frames composed of thin-walled beam elements.[5,6] Equation (4), in conjunction with usual iterative procedures[7] for fully-stressing truss elements constrained by tensile and compressive yield stresses, constitutes a complete algorithm.

Example

The iterative procedure noted above was used to design the nine-bar truss shown in Fig. 2. This truss was originally examined in Ref. 1 where a solution was obtained by considering two design variables per element. An improved alternate solution was obtained in Ref. 2 based on use of the set of variables A_i, and inequality constraints in the form of (3b).

Results of the iterative solution are compared with the solution from Ref. 2 in Table 1. It may be seen that both designs are essentially identical. The fact that the iterative solution has slightly lower weight than the "exact" solution from Ref. 2 is due to incomplete convergence of the latter, and minor overstressing of the former, as may be observed from the values shown in

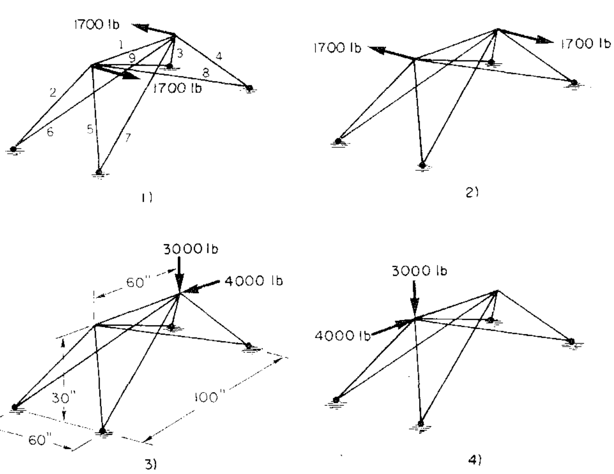

Fig. 2 Nine-bar truss subjected to four load conditions. ($E = 6.5 \times 10^6$ psi, specific weight = 0.065 lb/in^3, $\sigma_T = 30{,}000$ psi, $\sigma_c = 17{,}000$ psi, $K = 0.4$).

Table 1 Design parameters for truss of Fig. 2

Component	A, in.^{2}		D/t	
	Ref. 2	Iterative solution	Ref. 2	Iterative solution
1	0.0833	0.0511	209.8	268.1
2, 3, 4, 5	0.0740	0.0717	173.9	176.8
6, 7, 8, 9	0.1921	0.1972	208.5	205.8
Weight, lbs.	5.7528	5.7174	5.7528	5.7174

Table 2 Component stresses for final design of truss of Fig. 2 (iterative solution)

Component	σ_{Allow}, ksi	σ_{ij}, ksi			
		Load condition			
		1	2	3	4
1	−9.710	0	0	−9.719	−9.719
2	−14.707	13.409	−13.409	−1.624	−14.709
3	−14.707	−13.409	13.409	−14.709	−1.624
4	−14.707	13.409	−13.409	−14.709	−1.624
5	−14.707	−13.409	13.409	−1.624	−14.709
6	−12.634	−3.598	3.598	−12.633	1.140
7	−12.634	3.598	−3.598	−12.633	1.140
8	−12.634	−3.598	3.598	1.140	−12.633
9	−12.634	3.598	−3.598	1.140	−12.633

Table 2. Computation time required to obtain this iterative solution on an IBM 360/91 was on the order of 0.8 sec.

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Transverse Vibrations Related to Stability of Curved Anisotropic Plates

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Introduction

THIS Note investigates the effect of inplane compressive loading on the natural frequencies of vibration of 45° angle-ply and cross-ply laminated composite plates. As shown by Lurie[1] for flat isotropic plates, the load, sufficiently large so as to cause the lowest natural frequency to be reduced to zero, corresponds to the critical buckling load. Hence, by this method, buckling information has been obtained for both lamina configurations as well. In addition, the effects of slight initial curvature and of various boundary conditions on frequency and buckling load are studied.

Analytical buckling investigations of laminated plates appear somewhat limited; however, there does exist work in this area. Typically, Ashton and Waddoups[2] have investigated the buckling of symmetrically layered angle-ply plates with loaded edges clamped and unloaded edges simply-supported. Whitney and Leissa[3,4] have investigated buckling of unsymmetrically layered angle-ply plates with two types of hinge boundaries. To date it appears no analytical buckling information is available for the cross-ply case.

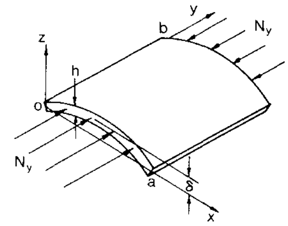

Fig. 1 Single constant curvature representation and inplane loading.

This study is restricted to those laminates symmetrically layered about the midplane or unsymmetrically layered but composed of a sufficient number of layers (usually 6 or more) so that the bending-extensional coupling present from unsymmetrical layering is eliminated. Without this restriction, buckling would not be characterized by a bifurcation but rather as a nonlinear (large displacement) response,[5] thus casting some doubt as to the validity of the results obtained from a linear analysis. In addition, the inplane loading is uniaxial and parallel to the axis of symmetry as shown in Fig. 1.

Analysis

Since the problem does not possess a closed form solution, the Ritz energy method was employed. The procedure, well established in the literature, requires the total energy expression expressed as a function of displacements. The strain-displacement relations used in this regard[6] are

$$\begin{aligned} \varepsilon_x &= u_{,x}{}^{o} - ww_{o,xx} - zw_{,xx} \\ \varepsilon_y &= v_{,y}{}^{o} - ww_{o,yy} - zw_{,yy} \\ \gamma_{xy} &= u_{,y}{}^{o} + v_{,x}{}^{o} - 2ww_{o,xy} - 2zw_{,xy} \end{aligned} \tag{1}$$

where u^o, v^o are midplane tangential displacements in the x, y directions, respectively, and w is the transverse displacement.

The initial shape, w_o, is assumed to be of the form $w_o = 4\delta(x/a - x^2/a^2)$ where δ is the maximum initial rise. With this form, a constant initial curvature, $w_{o,xx} = -8\delta/a^2$, is obtained. Note also that in (1) $w_{o,yy} = w_{o,xy} = 0$.

The strain energy expression for the plane stress situation assumed here is given by

$$\begin{aligned} U = \frac{1}{2}\int_o^a\int_o^b [&A_{11}(u_{,x}{}^{o} - ww_{o,xx})^2 + 2A_{12}(u_{,x}{}^{o} - ww_{o,xx})v_{,y}{}^{o} + \\ &2A_{16}(u_{,x}{}^{o} - ww_{o,xx})(u_{,y}{}^{o} + v_{,x}{}^{o}) + A_{22}(u_{,y}{}^{o})^2 + \\ &2A_{26}(u_{,y}{}^{o} + v_{,x}{}^{o})v_{,y}{}^{o} + A_{66}(u_{,y}{}^{o} + v_{,x}{}^{o})^2 + D_{11}w_{,xx}{}^2 + \\ &2D_{12}w_{,xx}w_{,yy} + 4D_{16}w_{,xx}w_{,xy} + D_{22}w_{,yy}{}^2 + \\ &4D_{26}w_{,xy}w_{,yy} + 4D_{66}w_{,xy}{}^2 + N_y w_{,y}{}^2]\,dx\,dy \end{aligned} \tag{2}$$

where A_{ij}, D_{ij} are constitutive coefficients, as defined in Ref. 3, and N_y is the inplane force per unit length as shown in Fig. 1. The kinetic energy, T, is given by

$$T = \frac{1}{2}\int_o^a\int_o^b \rho h[(u_{,t}{}^{o})^2 + (v_{,t}{}^{o})^2 + (w_{,t})^2]\,dx\,dy \tag{3}$$

Table 1 Boundary conditions

	At $x = 0, a$	At $y = 0, b$
HFT	$w = u^o = M_x = N_{xy} = 0$	$w = u^o = M_y = N_{xy} = 0$
HFN	$w = v^o = M_x = N_x = 0$	$w = u^o = M_y = N_y = 0$
HR	$w = u^o = v^o = M_x = 0$	$w = u^o = v^o = M_y = 0$
CL	$w = w_{,x} = u^o = v^o = 0$	$w = w_{,y} = u^o = v^o = 0$

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